Numerical analysis of a planar motion

GeoGebra as a tool of investigation

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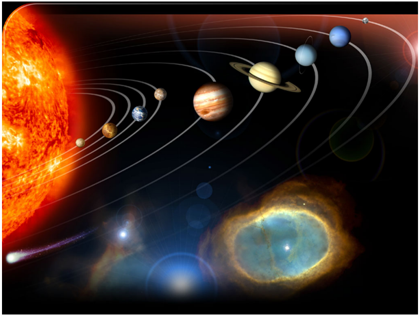
OBJECTIVE OF THE LECTURE

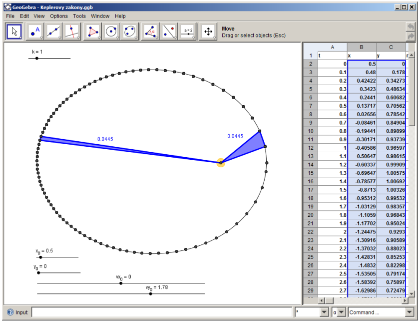
To present GeoGebra as a tool that enables students to **investigate the mathematical background of various real-world phenomena** and to produce their geometrical and mathematical models.

To show, through a couple of examples, that GeoGebra, thanks to its unique combination of the tools Spreadsheet View, Algebra View and Drawing Pad, represents a **powerful tool that enables us to simply and naturally perform numerical iterative computation and to plot its results**.

PLANETARY MOTION

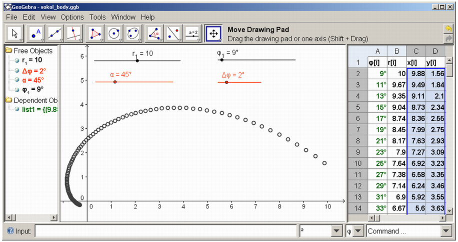
Numerical approximation of a motion of a planet around the sun





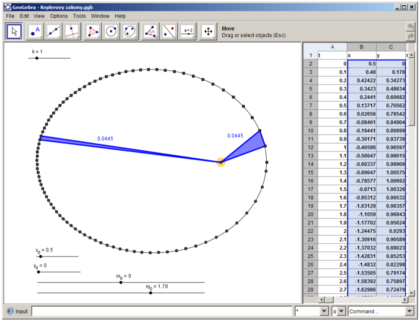
FALCON'S ATTACK

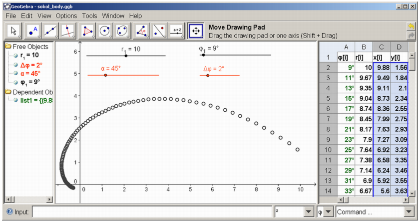
Approximation of the trajectory along which the Peregrine Falcon approaches its prey from a great distance.



GeoGebra features

What do the selected examples have in common?



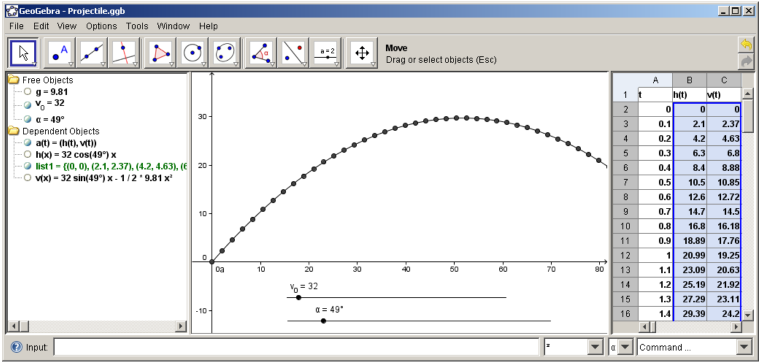


- Dynamic connection of Views.
- Iterative computation via the Spreadsheet.

DYNAMIC CONNECTION OF VIEWS

Algebra View - Spreadsheet View - Drawing Pad

Example: A projectile motion under gravity in a vacuum with the angle of elevation α and the initial velocity v_0.

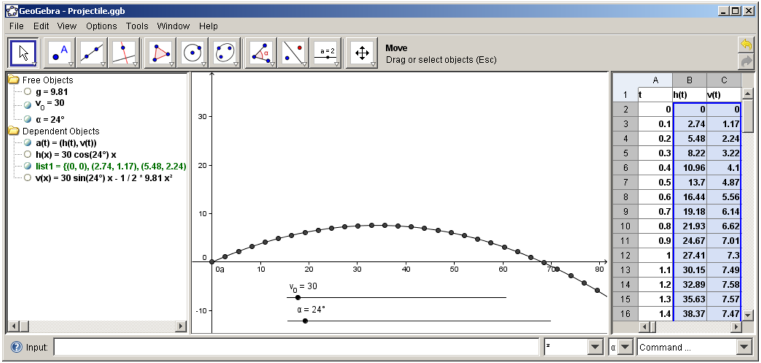


$$v_0 = 32\,\mathrm{m}\cdot\mathrm{s}^{-1},\ \alpha = 49^\circ$$

Dynamic connection of Views

Algebra View - Spreadsheet View - Drawing Pad

Example: A projectile motion under gravity in a vacuum with the angle of elevation α and the initial velocity v_0.

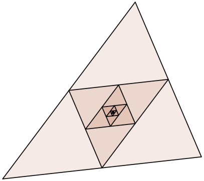


$$v_0 = 30\,\mathrm{m}\cdot\mathrm{s}^{-1},\ \alpha = 24^\circ$$

Iterative computation via the Spreadsheet

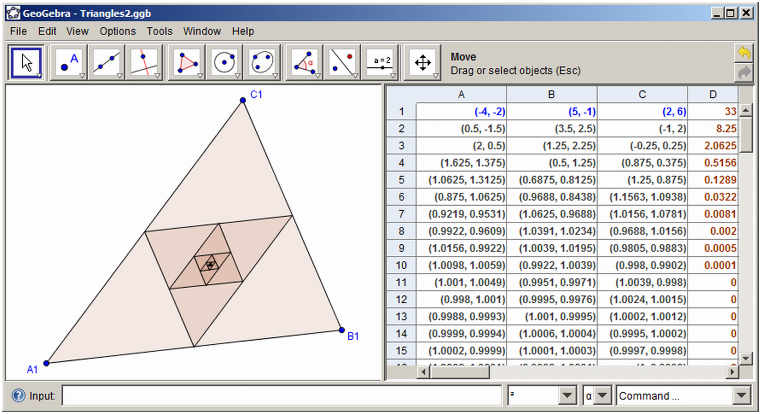
Thanks to the relative addresses of the cells the Spreadsheet enables us to perform iterative computation - the values computed on one line of a table can be used as input values for computation on the subsequent line.

Example: Nested triangles. Plot a sequence of triangles so that the vertices of the successive triangle are the midpoints of the sides of its predecessor.



	A	B	C	D
1	`(-4,-2)`	`(5,-1)`	`(2,6)`	`Polygon[A1,B1,C1]`
2	`(A1+B1)/2`	`(B1+C1)/2`	`(C1+A1)/2`	`Polygon[A2,B2,C2]`
3	`(A2+B2)/2`	`(B2+C2)/2`	`(C2+A2)/2`	`Polygon[A3,B3,C3]`
4	...	...	...	...
5	...	...	...	...

Iterative computation via the Spreadsheet



Planetary motion

Kepler's laws of planetary motion

From Wikipedia, the free encyclopedia

In astronomy, **Kepler's laws** give a description of the motion of planets around the Sun.

Kepler's laws are:

1. The orbit of every planet is an ellipse with the Sun at one of the two foci.
2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.[1]
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Kepler's laws are strictly only valid for a lone (not affected by the gravity of other planets) zero-mass object orbiting the Sun; a physical impossibility. Nevertheless, Kepler's laws form a useful starting point to calculating the orbits of planets that do not deviate too much from these restrictions.

Isaac Newton solidified Kepler's laws by showing that they were a natural consequence of his inverse square law of gravity with the limits set in the previous paragraph. Further, Newton extended Kepler's laws in a number of important ways such as allowing the calculation of orbits around other celestial bodies.

The past Johannes Kepler published his first two laws in 1609, having found them by analyzing the astronomical observations of Tycho Brahe.[2] Kepler did not discover his third law until many years later, and it was published in

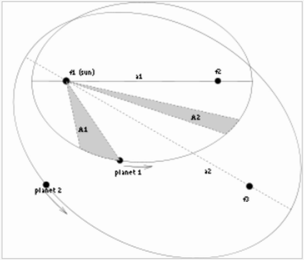


Figure 1: Illustration of Kepler's three laws with two planetary orbits. (1) The orbits are ellipses, with focal points f_1 and f_2 for the first planet and f_1 and f_3 for the second planet. The Sun is placed in focal point f_1. (2) The two shaded sectors A_1 and A_2 have the same surface area and the time for planet 1 to cover segment A_1 is equal to the time to cover segment A_2. (3) The total orbit times for planet 1 and planet 2 have a ratio $a_1^{3/2} : a_2^{3/2}$.

ASSIGNMENT

Feynman, R.P., Leighton, R.B., Sands, M. *The Feynman Lectures on Physics*, Volume 1, chapter "Newton's Laws of Dynamics", page 9-6, first published in 1964

"... **can we analyze the motion of a planet around the sun? Let us see whether we can arrive at an approximation to an ellipse for the orbit**."

Initial conditions:

- A planet starts at a given place $X(0)$ with a given velocity $\vec{v}(0)$.
- Newton's laws of motion and Newton's law of gravitation are valid.

Feynman's solution

Basic steps:

- Convenient choice of physical constants to simplify the computation (to get $\kappa M = 1$ in $F = \kappa \frac{Mm}{r^2}$).
- Application of Newton's Second Law and the law of gravitation (to compute the acceleration)
- Derivation of a relation between two successive positions $X(t_i)$, $X(t_{i+1})$ of a planet during its motion around the sun.
- Determination of the initial position $X(0) = [x(0), y(0)]$ and velocity $\vec{v}(0) = (v_x(0), v_y(0))$ of the planet.
- Sequential computation of the consecutive positions of the planet.

Sequential computation of the consecutive positions

$$t_{i+1} = t_i + \Delta t,$$

$$x(t_{i+1}) = x(t_i) + \Delta t v_x(t_i),$$
$$y(t_{i+1}) = y(t_i) + \Delta t v_y(t_i),$$

$$v_x(t_{i+1}) = v_x(t_i) + \Delta t a_x(t_i),$$
$$v_y(t_{i+1}) = v_y(t_i) + \Delta t a_y(t_i),$$

$$a_x(t_i) = -\frac{x(t_i)}{r(t_i)^3},$$

$$a_y(t_i) = -\frac{y(t_i)}{r(t_i)^3}$$

Table 9–2

Solution of $dv_x/dt = -x/r^3$, $dv_y/dt = -y/r^3$, $r = \sqrt{x^2 + y^2}$.

Interval: $\epsilon = 0.100$

Orbit $v_y = 1.63$ $v_x = 0$ $x = 0.5$ $y = 0$ at $t = 0$

t	x	v_x	a_x	y	v_y	a_y	r	$1/r^3$
0.0	0.500		−4.00	0.000		0.00	0.500	8.000
		−0.200			1.630			
0.1	0.480		−3.68	0.163		−1.25	0.507	7.675
		−0.568			1.505			
0.2	0.423		−2.91	0.313		−2.15	0.526	6.873
		−0.859			1.290			
0.3	0.337		−1.96	0.442		−2.57	0.556	5.824
		−1.055			1.033			
0.4	0.232		−1.11	0.545		−2.62	0.592	4.81
		−1.166			0.771			
0.5	0.115		−0.453	0.622		−2.45	0.633	3.942
		−1.211			0.526			
0.6	−0.006		+0.020	0.675		−2.20	0.675	3.252

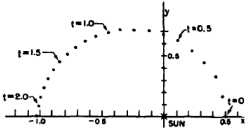


Fig. 9–6. The calculated motion of a planet around the sun.

Sequential computation of the consecutive positions

GeoGebra

File Edit View Options Tools Window Help

Move
Drag or select objects (Esc)

Algebra View

Free Objects
Dependent Objects

$$t_{i+1} = t_i + \Delta t,$$

$$x(t_{i+1}) = x(t_i) + \Delta t v_x(t_i),$$
$$y(t_{i+1}) = y(t_i) + \Delta t v_y(t_i),$$

$$v_x(t_{i+1}) = v_x(t_i) + \Delta t a_x(t_i),$$
$$v_y(t_{i+1}) = v_y(t_i) + \Delta t a_y(t_i),$$

$$a_x(t_i) = -\frac{x(t_i)}{r(t_i)^3},$$

$$a_y(t_i) = -\frac{y(t_i)}{r(t_i)^3}$$

Spreadsheet View

t	x	v_x	a_x	y	v_y	a_y	r	$1/r^3$
0.0	0.500		−4.00	0.000		0.00	0.500	8.000
		−0.200			1.630			
0.1	0.480		−3.68	0.163		−1.25	0.507	7.675
		−0.568			1.505			
0.2	0.423		−2.91	0.313		−2.15	0.526	6.873
		−0.859			1.290			
0.3	0.337		−1.96	0.442		−2.57	0.556	5.824
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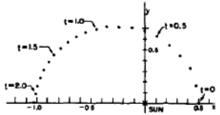
0.4	0.232		−1.11	0.545		−2.62	0.592	4.81
		−1.166			0.771			
0.5	0.115		−0.453	0.622		−2.45	0.633	3.942
		−1.211			0.526			
0.6	−0.006		+0.020	0.675		−2.20	0.675	3.252

Drawing Pad



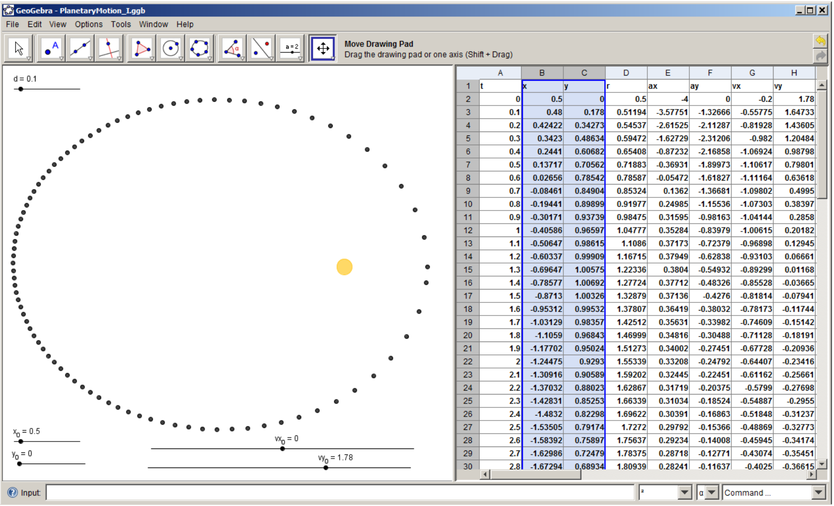
Input: Command ...

ITERATIVE COMPUTATION VIA THE SPREADSHEET

	A	B	C	D	E	F	G	H
1	t	x	y	r	ax	ay	vx	vy
2	0	x_0	y_0	sqrt($B2^2 + C2^2$)	$-B2/D2^3$	$-C2/D2^3$	$vx_0 + d/2$ E2	$vy_0 + d/2$ F2
3	A2 + d	B2 + d G2	C2 + d H2	sqrt($B3^2 + C3^2$)	$-B3/D3^3$	$-C3/D3^3$	G2 + d E3	H2 + d F3
4	A3 + d	B3 + d G3	C3 + d H3	sqrt($B4^2 + C4^2$)	$-B4/D4^3$	$-C4/D4^3$	G3 + d E4	H3 + d F4
5	...	...	...	...	...	...	...	...
6	...	...	...	...	...	...	...	...

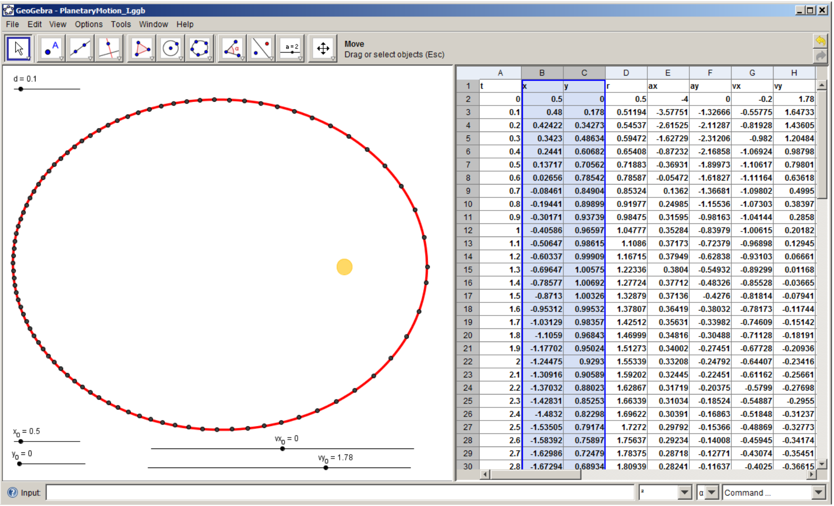
	A	B	C	D	E	F	G	H
1	t	x	y	r	ax	ay	vx	vy
2	0	0.5	0	0.5	-4	0	-0.18	1.78
3	0.1	0.4838	0.1602	0.50963	-3.65503	-1.21029	-0.50895	1.67107
4	0.2	0.43799	0.3106	0.53694	-2.82931	-2.00636	-0.76359	1.4905
5	0.3	0.36927	0.44474	0.57806	-1.9117	-2.30241	-0.93564	1.28328
6	0.4	0.28506	0.56024	0.62859	-1.14772	-2.25563	-1.03894	1.08028

SOLUTION IN GEOGEBRA



	A	B	C	D	E	F	G	H
1	t	x	y	r	ax	ay	vx	vy
2	0	0.5	0	0.5	-4	0	-0.2	1.78
3	0.1	0.48	0.178	0.51194	-3.57751	-1.32666	-0.55775	1.64733
4	0.2	0.42422	0.34273	0.54537	-2.61525	-2.11287	-0.81928	1.43605
5	0.3	0.3423	0.48634	0.59472	-1.62729	-2.31206	-0.982	1.20484
6	0.4	0.2441	0.60682	0.65408	-0.87232	-2.16858	-1.06924	0.98798
7	0.5	0.13717	0.70562	0.71883	-0.36931	-1.89973	-1.10617	0.79801
8	0.6	0.02656	0.78542	0.78587	-0.05472	-1.61827	-1.11164	0.63618
9	0.7	-0.08461	0.84904	0.85324	0.1362	-1.36681	-1.09802	0.4995
10	0.8	-0.19441	0.89899	0.91977	0.24985	-1.15536	-1.07303	0.38397
11	0.9	-0.30171	0.93739	0.98475	0.31595	-0.98163	-1.04144	0.2858
12	1	-0.40586	0.96597	1.04777	0.35284	-0.83979	-1.00615	0.20182
13	1.1	-0.50647	0.98615	1.1086	0.37173	-0.72379	-0.96898	0.12945
14	1.2	-0.60337	0.99909	1.16715	0.37949	-0.62838	-0.93103	0.06661
15	1.3	-0.69647	1.00575	1.22336	0.3804	-0.54932	-0.89299	0.01168
16	1.4	-0.78577	1.00692	1.27724	0.37712	-0.48326	-0.85528	-0.03665
17	1.5	-0.8713	1.00326	1.32879	0.37136	-0.4276	-0.81814	-0.07941
18	1.6	-0.95312	0.99532	1.37807	0.36419	-0.38032	-0.78173	-0.11744
19	1.7	-1.03129	0.98357	1.42512	0.35631	-0.33982	-0.74609	-0.15142
20	1.8	-1.1059	0.96843	1.46999	0.34816	-0.30488	-0.71128	-0.18191
21	1.9	-1.17702	0.95024	1.51273	0.34002	-0.27451	-0.67728	-0.20936
22	2	-1.24475	0.9293	1.55339	0.33208	-0.24792	-0.64407	-0.23416
23	2.1	-1.30916	0.90589	1.59202	0.32445	-0.22451	-0.61162	-0.25661
24	2.2	-1.37032	0.88023	1.62867	0.31719	-0.20375	-0.5799	-0.27698
25	2.3	-1.42831	0.85253	1.66339	0.31034	-0.18524	-0.54887	-0.2955
26	2.4	-1.4832	0.82298	1.69622	0.30391	-0.16863	-0.51848	-0.31237
27	2.5	-1.53505	0.79174	1.7272	0.29792	-0.15366	-0.48869	-0.32773
28	2.6	-1.58392	0.75897	1.75637	0.29234	-0.14008	-0.45945	-0.34174
29	2.7	-1.62986	0.72479	1.78375	0.28718	-0.12771	-0.43074	-0.35451
30	2.8	-1.67294	0.68934	1.80939	0.28241	-0.11637	-0.4025	-0.36615

SOLUTION IN GEOGEBRA - 1ST KEPLER'S LAW



	A	B	C	D	E	F	G	H
1	t	x	y	r	ax	ay	vx	vy
2	0	0.5	0	0.5	-4	0	-0.2	1.78
3	0.1	0.48	0.178	0.51194	-3.57751	-1.32666	-0.55775	1.64733
4	0.2	0.42422	0.34273	0.54537	-2.61525	-2.11287	-0.81928	1.43605
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8	0.6	0.02656	0.78542	0.78587	-0.05472	-1.61827	-1.11164	0.63618
9	0.7	-0.08461	0.84904	0.85324	0.1362	-1.36681	-1.09802	0.4995
10	0.8	-0.19441	0.89899	0.91977	0.24985	-1.15536	-1.07303	0.38397
11	0.9	-0.30171	0.93739	0.98475	0.31595	-0.98163	-1.04144	0.2858
12	1	-0.40586	0.96597	1.04777	0.35284	-0.83979	-1.00615	0.20182
13	1.1	-0.50647	0.98615	1.1086	0.37173	-0.72379	-0.96898	0.12945
14	1.2	-0.60337	0.99909	1.16715	0.37949	-0.62838	-0.93103	0.06661
15	1.3	-0.69647	1.00575	1.22336	0.3804	-0.54932	-0.89299	0.01168
16	1.4	-0.78577	1.00692	1.27724	0.37712	-0.48326	-0.85528	-0.03665
17	1.5	-0.8713	1.00326	1.32879	0.37136	-0.4276	-0.81814	-0.07941
18	1.6	-0.95312	0.99532	1.37807	0.36419	-0.38032	-0.78173	-0.11744
19	1.7	-1.03129	0.98357	1.42512	0.35631	-0.33982	-0.74609	-0.15142
20	1.8	-1.1059	0.96843	1.46999	0.34816	-0.30488	-0.71128	-0.18191
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23	2.1	-1.30916	0.90589	1.59202	0.32445	-0.22451	-0.61162	-0.25661
24	2.2	-1.37032	0.88023	1.62867	0.31719	-0.20375	-0.5799	-0.27698
25	2.3	-1.42831	0.85253	1.66339	0.31034	-0.18524	-0.54887	-0.2955
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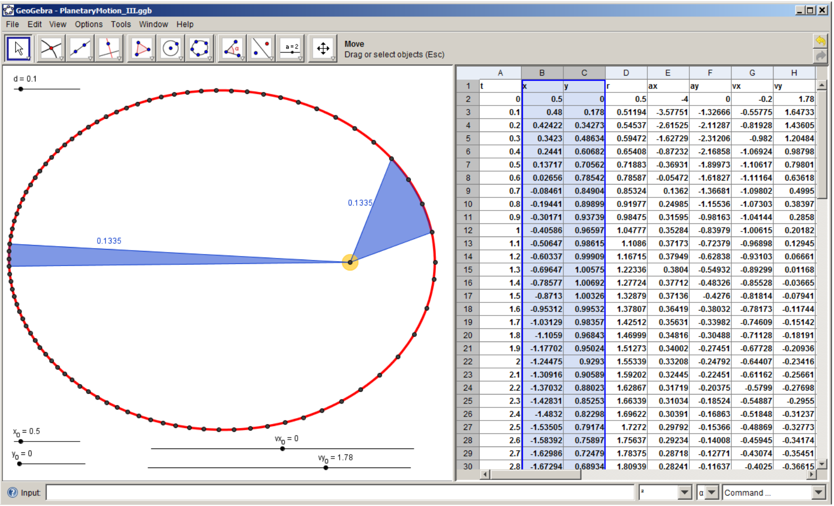
27	2.5	-1.53505	0.79174	1.7272	0.29792	-0.15366	-0.48869	-0.32773
28	2.6	-1.58392	0.75897	1.75637	0.29234	-0.14008	-0.45945	-0.34174
29	2.7	-1.62986	0.72479	1.78375	0.28718	-0.12771	-0.43074	-0.35451
30	2.8	-1.67294	0.68934	1.80939	0.28241	-0.11637	-0.4025	-0.36615

SOLUTION IN GEOGEBRA - 2ND KEPLER'S LAW



	A	B	C	D	E	F	G	H
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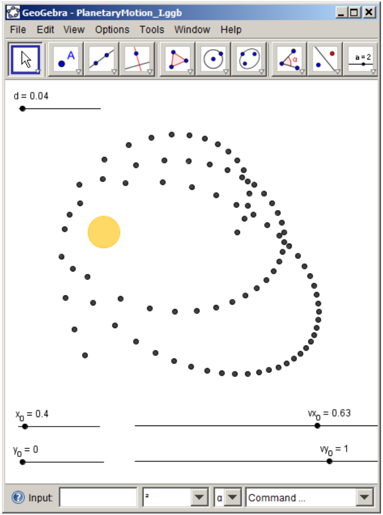
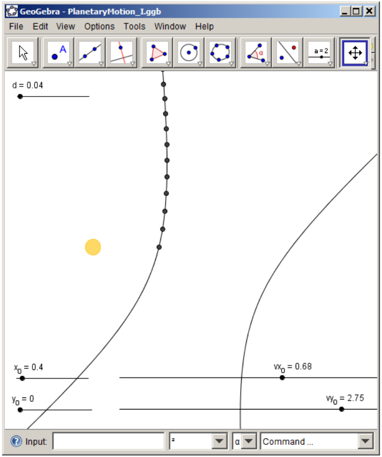
11	0.9	-0.30171	0.93739	0.98475	0.31595	-0.98163	-1.04144	0.2858
12	1	-0.40586	0.96597	1.04777	0.35284	-0.83979	-1.00615	0.20182
13	1.1	-0.50647	0.98615	1.1086	0.37173	-0.72379	-0.96898	0.12945
14	1.2	-0.60337	0.99909	1.16715	0.37949	-0.62838	-0.93103	0.06661
15	1.3	-0.69647	1.00575	1.22336	0.3804	-0.54932	-0.89299	0.01168
16	1.4	-0.78577	1.00692	1.27724	0.37712	-0.48326	-0.85528	-0.03665
17	1.5	-0.8713	1.00326	1.32879	0.37136	-0.4276	-0.81814	-0.07941
18	1.6	-0.95312	0.99532	1.37807	0.36419	-0.38032	-0.78173	-0.11744
19	1.7	-1.03129	0.98357	1.42512	0.35631	-0.33982	-0.74609	-0.15142
20	1.8	-1.1059	0.96843	1.46999	0.34816	-0.30488	-0.71128	-0.18191
21	1.9	-1.17702	0.95024	1.51273	0.34002	-0.27451	-0.67728	-0.20936
22	2	-1.24475	0.9293	1.55339	0.33208	-0.24792	-0.64407	-0.23416
23	2.1	-1.30916	0.90589	1.59202	0.32445	-0.22451	-0.61162	-0.25661
24	2.2	-1.37032	0.88023	1.62867	0.31719	-0.20375	-0.5799	-0.27698
25	2.3	-1.42831	0.85253	1.66339	0.31034	-0.18524	-0.54887	-0.2955
26	2.4	-1.4832	0.82298	1.69622	0.30391	-0.16863	-0.51848	-0.31237
27	2.5	-1.53505	0.79174	1.7272	0.29792	-0.15366	-0.48869	-0.32773
28	2.6	-1.58392	0.75897	1.75637	0.29234	-0.14008	-0.45945	-0.34174
29	2.7	-1.62986	0.72479	1.78375	0.28718	-0.12771	-0.43074	-0.35451
30	2.8	-1.67294	0.68934	1.80939	0.28241	-0.11637	-0.4025	-0.36615

EXPERIMENTATION WITH THE SOLUTION IN GEOGEBRA

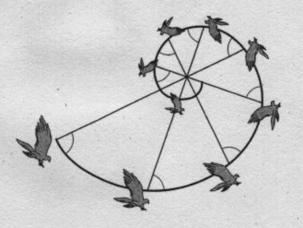




Falcon's attack trajectory

Livio, M. *The Golden Ratio: the story of phi, the world's most astonishing number*. 1st ed., Broadway Books, USA, 2003, page 118.

Peregrine Falcon, one of the fastest birds, traces the logarithmic spiral when it is approaching its prey which it has sighted from a great distance.



Eye's anatomy versus Aerodynamics

Tucker, V. A., Duke University, USA:

The spiral trajectory of the Falcon's attack is the result of a compromise between its desire for fast action and the reality of its eyes' anatomy.

EYE'S ANATOMY VERSUS AERODYNAMICS

Tucker, V. A., Duke University, USA:

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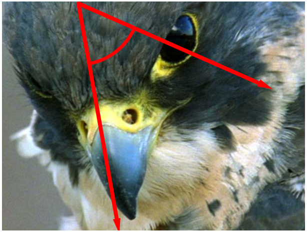


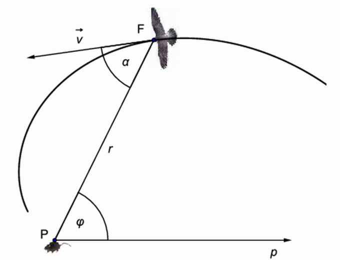
Peregrine Falcon's vision
The eyes' anatomy enables the use of the eye's most acute vision only in a specific line of sight which deviates considerably (about 40 degrees) from the frontal line.

ASSIGNMENT

Let's try to approximate the trajectory of the movement of the Falcon during its attack.

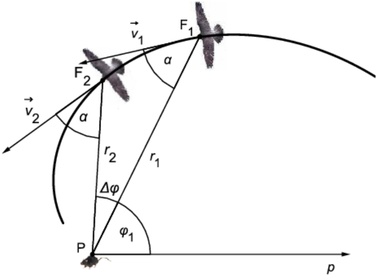
Initial conditions:

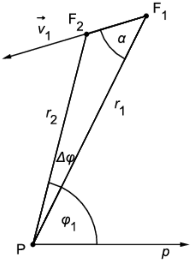
- Line of sight deviated α degrees to the left or right from the frontal direction (i.e. direction of the Falcon's instantaneous velocity $\vec{v}$).
- Permanent visibility of the prey.



SOLUTION

We will derive the relation between the polar coordinates of two consecutive approximate positions of the Falcon.





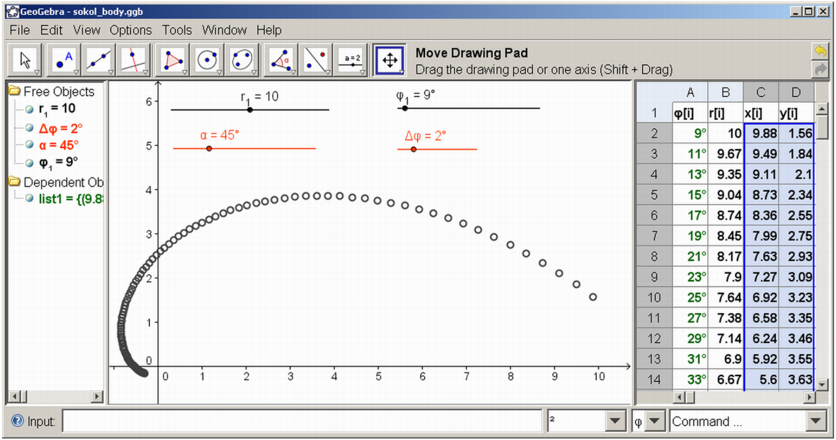
$$(r_{n+1}, \varphi_{n+1}) = \left(r_n \frac{\sin\alpha}{\sin(\alpha + \triangle\varphi)}, \varphi_n + \triangle\varphi \right)$$

ITERATIVE COMPUTATION VIA THE SPREADSHEET

$$(r_{n+1}, \varphi_{n+1}) = \left(r_n \frac{\sin \alpha}{\sin(\alpha + \triangle\varphi)}, \varphi_n + \triangle\varphi \right)$$

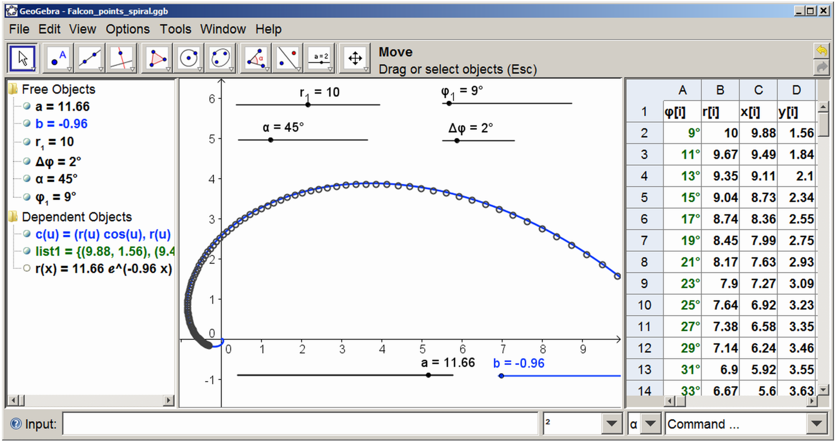
	A	B	C	D
1	`"`φ`[i]"`	`"r[i]"`	`"x[i]"`	`"y[i]"`
2	φ_1	r_1	$B2 \cdot \cos(A2)$	$B2 \cdot \sin(A2)$
3	$A2 + \triangle\varphi$	$B2 \cdot \sin(\alpha) / \sin(\alpha + \triangle\varphi)$	$B3 \cdot \cos(A3)$	$B3 \cdot \sin(A3)$
4	$A3 + \triangle\varphi$	$B3 \cdot \sin(\alpha) / \sin(\alpha + \triangle\varphi)$	$B4 \cdot \cos(A4)$	$B4 \cdot \sin(A4)$
5	$A4 + \triangle\varphi$	$B4 \cdot \sin(\alpha) / \sin(\alpha + \triangle\varphi)$	$B5 \cdot \cos(A5)$	$B5 \cdot \sin(A5)$
6	...	...	...	...

Solution in GeoGebra

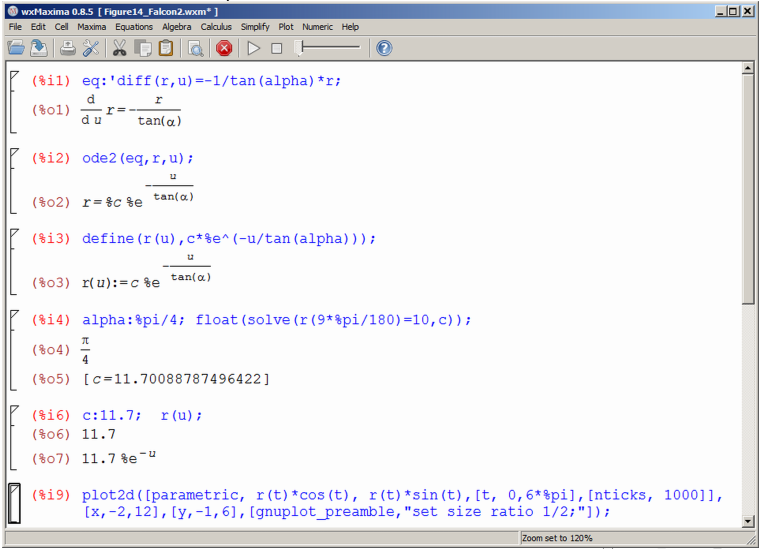


Equation of the spiral - the use of slider

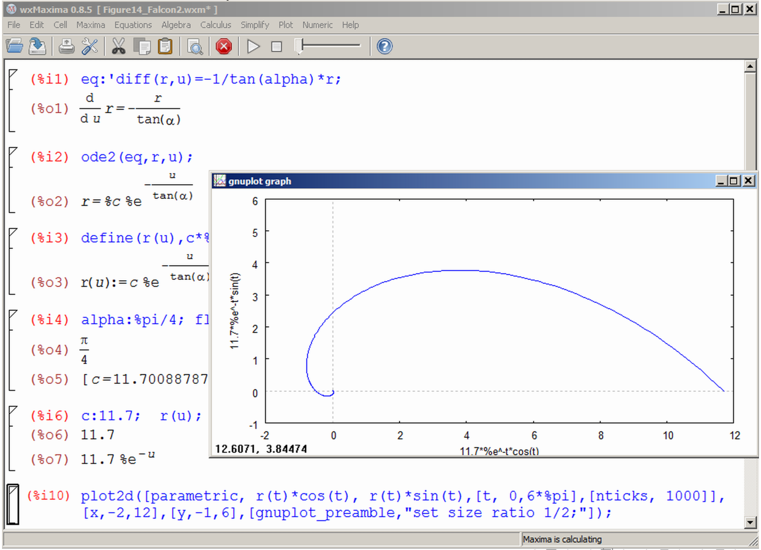
How the approximated curve (indicated by points) satisfy the equation of the logarithmic spiral $r(\varphi) = ae^{b\varphi}$?



Equation of the spiral - the use of wxMaxima (GeoGebra CAS)



Equation of the spiral - the use of wxMaxima (GeoGebra CAS)



CONCLUSION

Introduction of mathematics as a living and useful science.

Students have a chance to experience the usefulness of the mathematical knowledge they have learnt at school.